THE

MATHEMATICAL GAZETTE.

EDITED BY
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WITH THE CO-OPERATION OF
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The Mathematical Association.

THE Annual Meeting of the Mathematical Association was held at the London Day Training College, Southampton Row, W.C., on Saturday, 9th January, 1915, at 2.30 p.m.

The President, Sir George Greenhill, F.R.S., was in the chair.

The proceedings opened with the presentation of the following Report of the Council for 1914:

During the year 1914, 59 new members have been elected, and the number of members now on the Roll is 749. Of these 8 are honorary members, 34 are life members by composition, 46 are life members under the old rule and 661 are ordinary members. The number of associates is about 200.

The Council regret to have to record the deaths of Miss M. M. Jackson, Mr. Morgan Jenkins, Dr. J. S. Mackay, Professor G. M. Minchin, F.R.S., Mr. J. Urquhart, and Mr. W. Gallatly. Professor Minchin was President of the Association for the Improvement of Geometrical Teaching in the years 1889 and 1890; his membership began in 1883. Mr. Morgan Jenkins was for many years one of the Honorary Secretaries of the London Mathematical Society, and became a member of the A.I.G.T. in 1885. An obituary notice of Dr. J. S. Mackay appeared in the *Mathematical Gazette*, No. 111 (May, 1914). His membership of the A.I.G.T. began as far back as 1871, the year of its foundation.

During the past year the Council have issued a *Catalogue of Current Mathematical Journals, etc.*, with the names of the Libraries in which they may be found. The very exacting labour involved in the compilation of this catalogue was undertaken by Mr. Greenstreet. A copy of the catalogue has been sent to each of those who are already members of the Association,

but to the public and to those who may become members of the Association after the close of 1914, a charge of 2/6 will be made. Copies can be obtained from the publishers, Messrs. G. Bell & Sons, Ltd.

The election of the various Teaching Committees, under the new constitution, was held for the second time in the spring of 1914. The results of the election were published in the *Mathematical Gazette*, No. 112 (July, 1914). The number of members who took part in the election was 156. Those who were entitled to vote included all women members of the Association, but of the other members, only those who were actually engaged in teaching in the "Public," or "Other Secondary" Schools.

On 16th May, 1914, a Special Meeting of the Association was held for the purpose of considering the proposed alteration of Rule VI., which treats of the relations between the Association and its Local Branches and Associates. The new Rule, as approved by the meeting, was published in the *Mathematical Gazette*, No. 112.

In May the Council endeavoured to ascertain the wishes of the members as to the arrangements to be made in respect of our annual meetings. Three courses were suggested: (1) that our annual meetings should be held in conjunction with the Association of Public School Science Masters, (2) that the Association should take part in the Combined Education Conference promoted by the Teachers' Guild, and (3) that these two courses should be adopted alternately. The votes turned slightly in favour of the first course.

The Association of Public School Science Masters is not holding its meeting this month, but our own meeting is being held in order that the business necessitated by the Rules may be transacted.

Miss E. R. Gwatkin and Mr. A. W. Siddons now vacate their seats as ordinary members of the Council, and the members of the Association present at the annual meeting are asked to nominate and elect their successors.

Sir George Greenhill, F.R.S., retires at this meeting from the office of President. The Council, in the name of the Association, desire to record their deep sense of the debt which the Association owes to him for the services which he has rendered to it during the past two years. As Sir George Greenhill's successor, the Council have the pleasure of nominating Professor A. N. Whitehead, Sc.D., F.R.S., to be President for the years 1915 and 1916. They also nominate Sir George Greenhill to be an honorary member and a Vice-President of the Association, and Mr. A. W. Siddons to be a Vice-President.

The Council again desire to express their hearty appreciation

of Mr. Greenstreet's services as editor of the *Mathematical Gazette*. This year the debt to him has been largely increased by the very laborious and trying work which he voluntarily undertook in the compilation of the Catalogue of Periodicals.

The Council desire also to offer their thanks to the authorities of the London Day Training College for their kindness in affording accommodation for the numerous meetings which have been held in the College by the Association, its Council and its committees.

The Report was unanimously adopted. Professor A. N. Whitehead, Sc.D., F.R.S., was elected President for the years 1915 and 1916, and Miss H. M. Sheldon and Mr. G. W. Palmer were elected to fill the vacancies on the Council. The Treasurer presented his Report, and the following papers were then read:

"Mathematics in Artillery Science," by the President, Sir George Greenhill.

"The Teaching of Modern Analysis," by Dr. W. P. Milne.

Most teachers are agreed that the drawing up of a sound and practicable scheme of instruction on Modern Analysis for boys who have definitely decided to qualify as professional mathematicians is one of the most pressing needs of the hour. The present paper reviews the existing situation and suggests that Higher Algebra be divided into two sections, (1) Algebra and (2) Modern Analysis. In (1) would be treated all subjects not involving the discussion of Limits, such as Permutations and Combinations, Determinants, etc. In (2) would be discussed Irrational Numbers, Limits, Convergence of Series, etc. It is further suggested that the subjects under (2) should be treated much more numerically than at present, so that the pupil may get the opportunity of seeing what he is doing or attempting to do.

"Circles of Curvature," by Mr. A. Lodge.

An elementary method of finding the equations of the circles of curvature at a point, multiple or other, of a plane curve, whose equation is in rectangular co-ordinates.

"Practical Work in connection with Mathematics," by Mr. R. C. Fawdry.

A brief paper for the purpose of initiating a discussion on the following points:—How far arrangements are now made in Schools for doing laboratory work in connection with the teaching of Mathematics; the possibility of co-ordinating practical and theoretical work; suitable experiments.

REPORT OF THE GENERAL TEACHING COMMITTEE.

The past year has been one of comparative inactivity. The new Committee did not come into office until May, so that no work of importance was possible before the mid-summer vacation. Since that time, owing to the outbreak of the war, the conditions have not seemed suitable for entering upon any work of an important or controversial character. The Committee

is now considering what work it can most fittingly undertake under existing circumstances.

A sub-committee was appointed at the first meeting of the Committee to consider the whole question of the Teaching of Geometry, and it is now engaged upon drawing up a Report. The various Special Committees are also engaged upon various Reports. The Girls' Schools Committee is considering the whole question of the Teaching of Mathematics in Girls' Schools. The Public Schools Committee is engaged upon a Report on the Teaching of Arithmetic; the Other Secondary Schools Committee is considering the Teaching of Arithmetic and Mechanics in Secondary Schools. Reports on these questions will be ready shortly.

During the year the Committee has made representations to London University with respect to the Syllabuses in Mathematics at the Matriculation Examination of the University, but so far without success.

Mr. Siddons has been re-elected Chairman of the Committee.

P. Abbott,
Hon. Secretary.

REPORT OF THE PUBLIC SCHOOLS SPECIAL COMMITTEE.

Four meetings of the Committee have been held during the year. A circular has been sent to the Senior Mathematical Master of each of the Public Schools, asking whether he desires that all boys entering his school should be acquainted with the same method for the multiplication and division of decimals, and if so, whether he is in favour of the methods given in the circular. The investigation has been completed, and the result is being sent to the General Committee.

A Report on the Teaching of Arithmetic in Public Schools is being prepared. This will shortly be presented to the General Committee.

G. W. Palmer,
Hon. Secretary.

REPORT OF THE "OTHER SECONDARY SCHOOLS" SPECIAL COMMITTEE.

The Report of the "Other Secondary Schools" Special Committee was published in the *Mathematical Gazette* of January, 1914, page 231. Since that time the re-constituted Committee, which fully endorses the Report, has held only two meetings. In view of the fact that the London University has not yet modified its Matriculation Syllabus in Mathematics, the Committee feels that a wider circulation of the Report is most desirable.

The Director of Examinations for the Civil Service Commission expressed a desire to have the views of the Committee as to the suitability of syllabuses suggested in Elementary and Further Mathematics for an examination of boys leaving a Secondary School at the age of about 16. The Committee made certain detailed suggestions which were communicated to the Director of Examinations.

The Committee is preparing to consider the teaching in Secondary Schools of (i) *Arithmetic* and (ii) *Mechanics.*

W. J. Dobbs,
Hon. Secretary.

REPORT OF THE GIRLS' SCHOOLS SPECIAL COMMITTEE.

This Committee has met three times during the year, and the sub-committee appointed by it has met twice. It has elected as its Chairman, Miss Gwatkin, and as its Secretary, Miss Punnett. One member has been co-opted—Miss Waters of the Bromley County Secondary School.

The Committee is engaged in continuing the work begun by the preceding committee, namely, the preparation of a Report on the Teaching of Mathematics in Girls' Schools. It is intended that this Report shall include not only a fairly detailed scheme of work in mathematics for girls, both specialists and non-specialists, but also suggestions as to methods of teaching the subject.

Margaret Punnett,
Hon. Secretary.

PYTHAGORAS.

The *Mathematical Gazette* has commenced a series of articles, adapted for the use of school-teachers, on the great mathematicians of former days. No series of the kind could be complete were Pythagoras omitted, for it was he who raised mathematics to the rank of a science, made it a part of liberal education, and evolved a scheme on which it continued to be studied in Europe for more than 2000 years. But though his claim to inclusion is indisputable, the materials for describing his work are sadly lacking. We have a treatise by Aristotle on Pythagorean teaching, but the biography by the wife of Pythagoras is lost, and only fragments of the sketches by Aristoxenus, Dikaiarchus, and Philolaus exist, while the memoirs by Laertius Diogenes, Iamblichus, and Porphyry, on which we have mainly to rely for personal details, are late and include much that is palpably untrue. The reputation of Pythagoras far exceeded that of any of his contemporaries, and so great was it that in the credulous ages miracles and magical feats were freely attributed to him; later these additions tended to make men go to the other extreme, and doubt the truth of everything written about him. Within recent years, however, the available materials have been subjected to critical examination, and the researches of T. Gomperz of Vienna, J. Burnet of St. Andrews, G. J. Allman of Dublin, and P. Tannery of Paris enable us to speak with more confidence than was formerly possible.

What we know for certain about Pythagoras's life comes to little more than saying that he was born about 570 B.C., his father being Mnesarchus, a prosperous tradesman of Samos; that he was a contemporary of Polycrates, the Tyrant of that State; that he travelled widely, and acquired great fame for his learning and moral teaching; that finally he settled at Croton in South Italy, where he founded an Order, whose members obtained for a time paramount political power; and that on the development of bitter opposition to his School he retired to Metapontum, where he died about 501 B.C. To these bald statements we may add that his opponents at Croton finally triumphed, but that later his followers re-established themselves as a philosophical Society, and for a century or more profoundly influenced Greek thought.

There are, however, many traditions about Pythagoras, and while frankly recognizing that the evidence is not good, I give a somewhat fuller outline, credible and even probable, of his career. Subject to this caution, we may say that Pythagoras was born about 570 B.C., and came of Tyrrhenian stock. His father was a prosperous goldsmith and lapidary of Samos; his mother a clever woman, who, at his birth dedicated her boy to the service of the Gods. His father had extensive business connections in Asia Minor, and,

probably through these, the lad made the acquaintance of various philosophers of the Ionian School. On growing up he went to Egypt, taking with him letters of recommendation from Polycrates to Amasis the reigning Pharaoh, and by the aid of the latter monarch secured admission to the College of Priests at Memphis or Diospolis, where he mastered the secrets of Egyptian science and religion. Thence he went to Babylon, and there learnt something of Persian and Indian thought.

Pythagoras had a burning zeal for knowledge. We may be certain that he was attracted by the geometrical discoveries of the Ionian School, which included the earliest attempts to give general proofs of geometrical propositions covering all particular instances, as also by the formulae and arithmetical rules known to the Egyptians and Babylonians. But I suspect that he was even more interested in the systems of religion as expounded by Egyptian, Assyrian, Persian, Jew, and Indian priests, and in the philosophical speculations of Thales and his School on the nature and form of the world.

He returned to Samos, perhaps about 535 B.C., with a great reputation for piety and learning. It is said that about this time he visited Delphi, and secured the consent of the priests to various much-needed reforms which he suggested. If true, this is a striking testimony to his fame and influence. It may be from this incident that he was subsequently regarded as being specially under the protection of Apollo. About 529 B.C. he migrated to Sicily, whence he removed first to Tarentum and then to Croton.

It was at Croton that his life-work was done. Just before his arrival, that city had been defeated by its ancient rival, Sybaris. It was universally felt that reforms were needed. On this ground, so well prepared, the seed of his teaching bore rapid fruit. The high standard of conduct which he advocated attracted wide attention, and finally he was given an opportunity to explain his views to the Senate. With their consent he formed a Brotherhood, and its establishment was followed by a revival of public spirit, and a cleansing of public life. Probably the Senate wished to use for their own purposes his influence with the Commonalty, and deemed it cheaply secured by their recognition of his teaching. His Order, ruled absolutely by him, was primarily a religious fraternity, but had also educational, social, scientific, and ethical sides: it was open to women as well as men. Branches were founded in neighbouring cities. Our chief interest in it to-day arises from its scientific achievements. The influence of a great community holding itself aloof from the general mass of citizens inevitably led to bitter opposition, to which was added the personal resentment of those candidates for admission to it whom Pythagoras rejected. Somewhere about 501 B.C. Pythagoras died at Metapontum, a neighbouring city to which he had temporarily moved. Shortly afterwards there were popular risings against the Pythagoreans throughout Southern Italy, and a house at Croton in which the leading members had taken refuge was burnt, many of his followers losing their lives.

With this tragedy the political career of the Order ended. It was, however, re-established as a philosophical Society, and the chief thinkers of the time were either members of it, or like the mathematicians Hippocrates and Eudoxus (founders of the Schools of Athens and Cyzicus) and the philosophers, Socrates and Plato, were greatly influenced by its doctrines. We may take the work of the four men last mentioned as opening new chapters in the history of mathematics and philosophy, and as marking the end, early in the fourth century before Christ, of the original Pythagorean School.

It will help to fix our ideas about the time at which Pythagoras lived if I add that it is said that he was an intimate friend of Hermodamus, whose grandfather had known Homer, and that in Assyria he may well have met Daniel, whose career, as told in the Old Testament, is familiar to us. We have no real evidence as to the appearance of Pythagoras, but according

to tradition he was a tall, handsome, grave, bearded man, usually clad entirely in white save for a purple belt and purple headgear. Undoubtedly he was a persuasive speaker, and could sway popular audiences as well as small bodies of experts. The authority he exercised among his contemporaries in so many subjects and ways is evidence of exceptional powers: he was at once a prophet, statesman, philosopher, and man of science. The accounts of the constitution of his Order, the ranks of membership, the ceremonies of initiation and advancement, and its internal regulations, while not improbable in themselves, must be received with great caution.

Primarily Pythagoras was a religious teacher and philosopher, inculcating a stern system of morality, greatly superior to any system then known to the Greeks. For reasons which I outline below he held that the ultimate explanation of things rested on a knowledge of numbers and form, that thus the study of philosophy must necessarily be associated with the study of pure and applied mathematics, and that these latter also provided the best general mental training and discipline. I am not here concerned with his ethical or philosophical opinions, of which indeed our knowledge is vague. Even on the teaching of his School in science we cannot speak with absolute assurance, while the further fact that his instruction was entirely oral introduces the possibility that commentators have failed to distinguish his conclusions from those made later by his followers. Thus the precise extent of his discoveries is still a matter of opinion, but I think we may reasonably attribute to him the results I proceed to describe. That he created mathematics as a science there is no doubt.

We cannot say in what order Pythagoras made his discoveries, but I conjecture that he was led to the study of geometry and numbers through his researches in natural philosophy. Of these, his investigations on acoustics were the most important.

Throughout his life, Pythagoras was profoundly moved by music, and in his School he taught that it was one of the chief means for exciting and calming emotions—it purged the soul, said his followers, as medicine purges the body. By a happy chance he constructed or came across an instrument consisting of a string stretched over a vibrating board with a movable bridge by which the string could be divided into different lengths. To his delight he found that other things being equal the note given by the vibrating string depended only on its length, and that the lengths that gave a note, its fourth, its fifth, and its octave were in the ratio 6, 8, 9, and 12. Thus suddenly the whole of an intangible and artistic world seemed reduced to a question of numbers.

It was not unnatural that he should suspect that other physical phenomena were explicable by similar means. This suspicion was strengthened by his noticing that many facts—seasons, the months, the tides, day and night, sleep and waking, the pulse, breathing, etc.—are periodic. Hence he argued that whatever be their explanation, it must involve a consideration of numbers, and he is said to have wondered whether the regular rhythmic sequences of the world would not ultimately be found to be analogous to the regular breathing of animals.

He appears to have been confirmed in the idea that fortune had placed in his hands the key to the riddle of the universe by considering the manner in which geometricians think of points, lines, and planes. We arrive at these ideas by a process of abstraction, considering a plane as a boundary of a polyhedron, a line as a boundary of a polygon, and a point as a boundary of a line, but he went further and suggested, Aristotle tells us, that in fact points, lines, and planes are more real than the concrete forms or figures from which we obtain our conception of them. He identified a point with unity; and he associated a line (which is determined by two points) with the number 2; a plane (which is determined by three points) with the number 3; and a

solid body with the number 4. Thus geometry seemed reducible to numbers and form.

So far Pythagoras's work rested on a sound foundation, but without evidence he extended his conclusions to other subjects, and perhaps at last came to believe that everything is connected with or typified by a particular number, and that through that number alone (if we can find it) can knowledge of the thing itself be obtained. According to tradition, some of these guesses, connecting particular qualities and ideas with particular numbers, were that justice was associated with the number 4; that the cause of colour was to be discovered in the properties of the number 5; and that friendship and harmony were explicable by the octave. We gather from Aristotle that many of these notions were put forward by Pythagoras in the latter years of his life, and it is believed that some of the wilder ones were due to his successors. We may at any rate say that he went further in associating particular things with particular numbers than the evidence justified, and that the mistake was magnified and extended by his School.

It was inevitable that so acute a thinker should consider the explanation of the more obvious astronomical phenomena. The Ionian philosophers had thought of the earth as a disc-like body floating on water, but Pythagoras taught that the earth and moon were spheres. Whether he extended the conception to the sun and planets is uncertain, nor do we know the details of his views on the subject, but it seems probable that he sought to resolve all astronomical phenomena into the circular motion of spherical bodies. Even in this crude form the suggestion showed scientific imagination of a high order.

By way of parenthesis I should add that this conception was subsequently developed, and his successors taught that the centre of the celestial sphere was occupied by a perpetual fire round which circled nine spherical bodies, namely, the earth, the moon, the sun, the five planets then recognized, and the firmament. They explained the apparent daily motion from east to west of the sun and moon as being really due to the more rapid motion of the earth from west to east. In order to bring the number of the celestial bodies to the mystic number ten, it was supposed that another planet, known as the counter-earth, circled round the central fire below the earth, always concealed from view by our unvisited hemisphere. The distances of this decade of celestial bodies were assumed to be in musical progression, and their motions were described poetically as set to the music arising from their movements and known as the harmony of the spheres. This extension was subsequent to the time of Pythagoras, and the introduction of the counter-earth, for which no evidence existed, was justly condemned by Aristotle and other philosophers.

It will be seen from the above sketch that Pythagoras discussed some problems in acoustics, common periodic sequences, geometrical conceptions, and astronomical phenomena, and that his conclusions led him to seek the explanation of everything in the theories of geometry and numbers. Before his age all that was known in geometry consisted of facts obvious to every observer (such as that two intersecting circles cut in two points) and a few isolated theorems due to the Ionian School (such as that the angles at the base of an isosceles triangle are equal). Of the then extant knowledge of the theory of numbers we speak with less certainty, but most likely it included results, known only to the initiated, about certain fractions and series. He raised both subjects to the rank of sciences. His discussion of numbers has been superseded by other and better ways of dealing with the subject, but his treatment of geometry was adopted by Euclid, and still forms the basis of modern elementary text books on it. I now proceed to describe his work on these subjects, on which to-day rests his chief claim to distinction.

On his geometry I have little to add to what I have said elsewhere. He probably knew and taught the substance of what is contained in the first

two books of Euclid about parallels, triangles, and parallelograms, and was acquainted with a few other isolated theorems. He taught that magnitudes may be represented by lines, and hence the value of general geometrical propositions.

It is hardly necessary to say that we are unable to reproduce the whole body of Pythagoras's teaching on geometry, but we gather from the notes of Proclus on Euclid, and from occasional remarks by other writers, that it included the following propositions, and such others as are required to prove them. (i) It commenced with a number of statements about mathematical conceptions: one has been preserved in the definition of a point as unity having position. (ii) He showed that the plane space about a point can be completely filled by equilateral triangles, by squares, and by regular hexagons—results that must have been familiar wherever tiles of these shapes were in common use. (iii) He proved that the sum of the angles of a triangle is equal to two right angles (Euc. i. 32): and in the demonstration, which has been preserved, the results of the propositions, Euc. i. 13 and the first part of Euc. i. 29, are quoted. The proof is substantially the same as that given by Euclid, and it is most likely that the proofs there given of the two propositions last mentioned are also due to Pythagoras himself. (iv) He established the properties of right-angled triangles which are given in Euc. i. 47 and i. 48, and the first of these propositions has since always been definitely associated with his name. We do not know how he proved it, though we are told that his demonstration is not that given in Euclid's *Elements*, but it has been observed that the theorem follows at once from the results of Euc. ii. 2, vi. 4, and vi. 17, with all of which Pythagoras was acquainted. (v) He is credited with the discovery of the theorems, Euc. i. 44 and i. 45, on the description of a parallelogram equal to one and similar to another parallelogram: his successors were aware of the extension given in Euc. vi. 25. (vi) He gave a construction for finding the geometrical mean of two lines (Euc. ii. 14). (vii) Of the five regular solids inscribable in a sphere, he was acquainted with four, and gave constructions for them. Probably he held that the elements of which the material world is made are related to these solids. (viii) He appears to have regarded the theory of proportion as a branch of geometry, but we know nothing about his work on it, and very likely he did not discuss it generally. He may have known that similar polygons are to each other in the ratio of their homologous sides, but it seems more likely that the discovery of this and the properties of the pentagon and dodecahedron were made by his successors. We may be confident that he did not deal with the properties of the circle, though he taught that the circle was the most perfect of plane figures, and the sphere the most perfect of solids. Subsequent Greek geometry was built on the foundation thus laid by Pythagoras. No doubt the presentation of the subject was improved and made more logical and rigorous by Hippocrates and Euclid. But this does not destroy the admiration felt for the man who created the first systematic exposition of the subject.

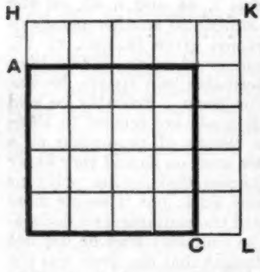
Perhaps the most far-reaching of his discoveries was that there were incommensurable numbers—a result which profoundly affected the subsequent development of Greek mathematics. The substance of the proof attributed to him may be put as follows. If a denotes the number of units of length in the diagonal of a square and b the number of units of length in a side, we have $a^2=2b^2$. If each of these numbers is a fraction or integer, we can, by multiplying by the L.C.M. of their denominators, bring this relation to a form in which a and b are integers. Further, we can strike out any factor common to a and b, and thus we may take it that a and b are prime to each other. Hence if one is even the other is odd. Now a^2 is even, hence a is even, and therefore b must be odd. But if a is even, let $a=2c$; then $4c^2=2b^2$, and therefore $b^2=2c^2$. Hence b^2 is even, and therefore b is even. Thus b

must at the same time be even and odd, which is impossible. Thus a and b must be incommensurable.

It follows that it may be impossible to measure a line in terms of a given unit of measurement. This led him to distrust demonstrations which rest on the possibility of making numerical measurements, for such measurements might not be accurate. Such magnitudes can, however, be represented geometrically by lines, and hence the absorption of theoretical arithmetic in geometry, which is characteristic of Greek mathematics. This geometrical treatment is illustrated by the proofs, probably due to Pythagoras, given in the second book of Euclid of propositions like $(a+b)^2=a^2+2ab+b^2$. I suspect that most of the proofs given by Euclid which involve the use of gnomons and their parts are Pythagorean in origin.

I proceed next to describe Pythagoras's researches on the science of numbers involving the investigation of properties of integers and their ratios. By way of preamble I should explain that he was not concerned with mercantile arithmetic, which necessarily formed part of the education of men of affairs, and in which the abacus was generally used. So too he was not concerned with problems in which there was a possibility of the introduction of incommensurable numbers: for these, geometrical methods were used. The particular problems with which he was concerned dealt with the properties of integers, the ratios of integers, special groups of inter-related integers, triangular integers, the factors of integers, and numbers in series.

Pythagoras commenced his discussion by pointing out the fundamental difference between geometry which deals with continuous quantities and arithmetic which deals with discrete quantities like integers. He divided all integers into even and odd, the odd numbers being termed gnomons. Curiously enough he went on to say that the odd integers were limited or finite and the even integers were unlimited or infinite: it is useless to speculate what he meant by this. An odd number, such as $2n+1$, was regarded as the difference of two square numbers, $(n+1)^2$ and n^2; and the sum of the gnomons from 1 to $2n+1$ was thus shown to be a square number, namely $(n+1)^2$; its square root was termed a side. Products of two numbers were called plane, and if a product had no exact square root it was termed an oblong. A product of three numbers was called a solid number, and, if the three numbers were equal, a cube. All this has obvious reference to geometry, and the opinion is confirmed by Aristotle's remark that when a gnomon is put round a square the figure remains a square though it is increased in dimensions. Thus, in the figure given above, in which n is taken equal to 4, the gnomon AKC (containing 9 cells) when put round the square AC (containing 4^2 cells) makes a square HL (containing 5^2 cells).



Pythagoras is said to have discussed ratio and proportion. We must take this in a very limited sense, for it is certain that he was ignorant of the classical Greek method invented more than a century later by Eudoxus, and set out in Euclid's *Elements*. I take it that Pythagoras's work was confined to the treatment of vulgar fractions, and dealt mainly with the reduction of fractions of the form a/b to a sum of fractions each of whose numerators is unity. Such problems played a considerable part in Egyptian priestly science. For example, the priests knew that 2/97 is the sum of 1/56, 1/679, and 1/776. The fact is that the early mathematicians tried to evade the difficulty of having to consider at the same time changes in both the numerator and denominator of fractions: they liked to have all the numerators or else all the denominators equal.

He dealt with particular groups of numbers, for instance, sets like 6, 8, 9, 12, which give the lengths of chords producing a note its fourth, its fifth, and its octave, and sets like $(2n^2+2n+1)$, $(2n^2+2n)$, and $(2n+1)$, which are proportional to the sides of certain right-angled triangles. Probably he did not know the more general expressions for similar sets like

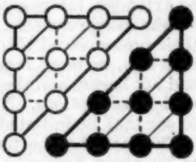
$$a,\ 2ab/(a+b),\ (a+b)/2,\ b, \text{ and } (m^2+n^2),\ 2mn,\ (m^2-n^2).$$

Pythagoras was acquainted with triangular numbers. It is not clear whether he also discussed polygonal numbers: his followers certainly did so. A triangular number represents the sum of a number of counters laid in rows on a plane: the bottom row containing n, and each succeeding row one less than the row before it; it is therefore equal to the sum of the series

$$n+(n-1)+(n-2)+\ldots+2+1.$$

Thus the triangular number corresponding to 4 is 10: this was known as the tetrad. This is the explanation of the language of Pythagoras in the well-known passage in Lucian where the merchant asks Pythagoras what he can teach him. Pythagoras replies, "I will teach you how to count." Merchant, "I know that already." Pythagoras, "How do you count?" Merchant, "One, two, three, four, ..." Pythagoras, "Stop! what you take to be four is ten, a perfect triangle, and the symbol of our oath." Pythagoras remarked that this mystic ten is the radix of notation generally adopted by mankind, and he pointed out that 10 is the lowest integer up to and including which there are an equal number of prime and composite numbers. To-day the tetrad appears in the arms of the Duchy of Cornwall and of the See of Worcester.

The Greek symbolism for numbers did not lend itself to operative uses, and was not employed for that purpose. Euclid, by choice, represented numbers by lines. The method by which Pythagoras proved properties of polygonal numbers was quite different, and rested on arranging pebbles (or beads strung on an abacus) in such a way as to give an ocular demonstration: it was still used by the Pythagoreans in the time of Eurytus, and is described by Aristotle. This method is not only interesting in the history of the subject, but offers demonstrations which are easily understood and are general in character. For instance the sum of two triangular numbers of the 4th order when expressed by pebbles arranged as shown in the annexed diagram is 4×5, *i.e.* 20, and thus each is equal to 10. A similar diagram shows that a triangular number of the nth order is equal to $n(n+1)/2$. Various similar numerical results which can be at once demonstrated in this way have been given by Lucas, for instance: eight times a triangular number of the order n augmented by unity is a square of the order $2n+1$; a pentagonal number of the order n is equal to n augmented by three times a triangular number of the order $n-1$; an hexagonal number of the order n is equal to n augmented by four times a triangular number of the order $n-1$.



As to the work of the Pythagoreans on the factors of numbers we know very little: they classified numbers by comparing them with the sum of their integral subdivisors or factors, calling a number excessive, perfect, or defective, according as it was greater than, equal to, or less than the sum of these subdivisors. Two numbers were called amicable if each was equal to the sum of the subdivisors of the other; they knew that 220 and 284 afforded an example of an amicable pair. Traces of their work on these subjects still appear in text-books. These researches may well have been expected to lead to valuable results, but in fact little came of them.

Pythagoras recognized arithmetic, geometric, and harmonic series. Perhaps

he learnt about the first two of these from Babylonian sources. He may have known how to sum an arithmetic series, but probably except for this he was not acquainted with any general theorems on such series.

The subjects described above were classified by Pythagoras as dealing with numbers absolute or Pythagorean arithmetic, numbers applied or music, magnitudes at rest or geometry, and magnitudes in motion or astronomy. This quadrivium was long considered as constituting a necessary course of study for a liberal education. All his disciples were required to study these subjects, and on them was based his scheme of education and philosophy.

Pythagoras and Thales were regarded as the founders of mathematics and philosophy, but the achievements of the former were much more remarkable than those of the older man, and their great value was recognized by the Greeks. Pythagoras made geometry, said Eudemus, into a liberal science, treated its principles from an abstract point of view, and investigated its theorems in a general and intellectual manner: he further discovered incommensurables, and the construction of the regular solids. The value of his arithmetical work was equally recognized. As a brilliant mathematician, the discoverer of the fundamental principles of acoustics, an acute student of astronomical phenomena, and a great ethical and religious teacher, his career stands unique. No one will deny him the distinction of having been one of the most original and influential thinkers of the ancient world. He has, said his contemporary, Heraclitus, when inveighing against him, practised research and inquiry more than all other men. Traces of his teaching still survive in our books, and the very name of mathematics, by which we call our science, is of his invention. It is true that the whole of his teaching was not sound, and that his theories were mixed with fanciful speculations, dangerous in the case of geometry and arithmetic, which are founded on inferences unconsciously made and common to all men, but almost necessarily fatal in the applied sciences, which can rest safely only on the results of conscious observation and experiment. We must also admit that he extended his results beyond what facts warranted: later generations of Greeks recognized this, and condemned his followers for adapting facts to suit their preconceived theories. But if finally he let his imagination run away with him, we must remember that he lived in a remote age, in which science had not begun, and there were no critics to exercise a sobering influence on his wilder assumptions. Notwithstanding these defects, it is not I think too much to say that he initiated the brilliant course of Greek philosophy and science.

The fact that he played a conspicuous part in politics led to opposition to his views during his life, but after his death the loftiness of his aims was generally admitted. According to Aristoxenus, his School gloried in the fact that they sought knowledge rather than wealth. There is reason to think that the classification of men into those that trade for profit, those that compete for honours, and those who observe, comes from Pythagoras himself, and that he held up to his disciples as the highest life a disinterested search for truth.

W. W. Rouse Ball.

THE ACHIEVEMENTS OF GREAT BRITAIN IN THE REALM OF MATHEMATICS.

(*Concluded.*)

VIII.

In England the sceptre of Mathematics passed from the hands of Wallis to the man who was to raise the exact sciences to a height hitherto unattained—Isaac Newton (1642–1727), that sovereign genius before whom the world of science will ever bow with the profoundest reverence.

He divides with Leibniz the honour of having taught a sound method for the rigorous treatment of the transcendental conception of "infinity," which became a potent weapon in the investigation of questions concerning the space in which we live, the forces at work in it, and the movements of which it is the theatre. But his name alone can be attached to the discovery of "universal gravitation," that fundamental concept which caused Lagrange to lament that, as there was but one universe, but one man could reveal its laws to humanity. Few will be found to contest the view that the *Principia* is the greatest production of the human mind. A new field of geometrical research was opened by his classification of plane cubics, and by showing the extraordinary fertility of the idea of "central projection" he laid the foundation of the Projective Geometry of to-day. His *Optics*, in spite of a point of view which has radically changed, is still a work that is consulted and quoted, and thus is worthy to-day of the admiration it excited at the time of its publication. And finally, in the *Arithmetica Universalis*, a didactic treatise published without the consent of the author, we have a volume which throws so much light on the subject that we cannot help remembering the ancient adage, that there is no subject, however trite and humble, in which the piercing glance of genius cannot detect some unexplored and brilliant features.

While the historic period during which Wallis reigned supreme was one of peace between England and the other countries of Europe, that in which Newton was king was the theatre of one of the longest and bitterest scientific controversies in the history of human thought. English mathematicians as one man entered the lists to guard Newton's rights of property in the Infinitesimal Calculus, while the whole of Germany rose to sustain the claim of Leibniz to the same discovery. Perhaps the definitive history of this interesting dispute has yet to be written. But it is universally agreed that there is no ground for the charges of theft or plagiary, as the two methods—the fluxional and the differential—while tending to the same goal are, if I may use the expression, radically different in their metaphysics.

The profound impression upon his contemporaries produced by the discoveries of Newton may be estimated by a glance at a list of his published works, and also by noting certain phenomena, really of a pathological character, due to the illusion that the new methods might be used successfully in the solution of any problem. Perhaps the most curious instance is that of a mathematician continually quoted, John Craig (?–1731), who, twelve years after the appearance of Newton's *opus magnum* brought out a volume entitled, *Theologiae Christianae Principia mathematica*, in which he maintained that evidence of the truth of the Christian religion must diminish in credibility with the square of the time—which implies that after the year 3150 not much trace of it will be left. You may be left to form your own opinion as to whether he was divinely inspired or a mere madman.

IX.

The profoundly beneficent influence of Newton on his time may be gauged by the output of those who openly announced themselves his disciples and followers. Considerations of time and discretion restrict me to the mention of men like James Stirling (1696–1770) and Patrick Murdoch (?–1744), who endeavoured to spread the ideas of the great mathematician and to fill up the gaps in his work. And I must confine myself to discussing three eminent investigators, who enabled Great Britain to retain the hegemony in mathematical physics which she

had acquired with the discovery of universal gravitation. I allude to Abraham Demoivre (1667–1754), Robert Cotes (1682–1716), and Brook Taylor (1685–1731).

To the first of the three the Theory of Probability was indebted for many improvements and for notable progress, and by the methods he taught he was able to solve a number of problems referred to this theory for solution. Nor will his name have passed unnoticed by those who have studied the theory of complex numbers, thanks to a fundamental application which is in constant use. Finally, to him belongs the honour of having discovered the formula which gives the power of a polynomial analogous to Newton's formula for the power of a binomial.

As for the reputation of the second of these men it is needless to do more than quote the words of the funeral eulogium pronounced by Newton: "If Mr. Cotes had lived, we should have learned something." He was one of the first to compile a list of formulae of integrals, and his posthumous *Harmonia Mensurarum* (1722) attests the extent and the success of his labours in this direction. In this volume we find what Robert Smith, his editor, calls the *pulcherrimum theorema*, completing the theorem of Demoivre mentioned above.

Not less distinguished is Brook Taylor, to whom we are indebted for notable progress in the Differential Calculus—it is needless to remind you of the formula which bears his name—the Integral Calculus (the integration of several classes of differential equations) and the Theory of Finite Differences. In another and quite different field he was able to aid certain thinkers of original genius—in that of Perspective. The new bases on which he placed this theory are so firm and solid that the lamented Wilhelm Fiedler, who knew nothing of Taylor's work, selected it when coordinating into one organic whole the various methods of Descriptive Geometry. Nor should we pass over in silence the fact that the work of the English geometer seemed to Cremona of such importance that he translated it into Italian for the use of his pupils.* A volume in the "Klassiker der exakten Wissenschaften" is a further sign that the value of Taylor's work is fully recognised.

And here it is to be placed on record that the new processes of investigation discovered by Newton and eagerly cultivated by his successors did not diminish the interest taken by the English in the methods of the geometers of old, for which Newton felt and expressed unbounded admiration. There is no necessity to furnish further proof of this than we find in the meritorious labours of Edmund Halley (1656–1742), to whom we are indebted, not only for the standard editions of Apollonius and Menelaus, but also for his successful endeavours to divine and to restore the lost writings of the immortal geometer of Perga. His example was followed by several illustrious successors, of whom we need but mention Robert Simson (1687–1768), Samuel Horsley (1733–1806), and William Wales (1734–1798).

And again, it must not be forgotten that Simson was the first to attack the solution of that extremely difficult and famous enigma in the history of Greek mathematics which consists in determining the meaning of the word "porism," and in restoring Euclid's three lost books. Moreover, by his judicious commentary in his admirable edition of Euclid he did excellent work in diffusing among his contemporaries a knowledge of that unrivalled treatise.

* Marco Uglieni (the anagram of Luigi Cremona), "I principii della 65 prospettiva lineare secondo Taylor," *Giornale di Matematiche*, t. iii. 1865.

X.

Among the commentaries (using the word in it widest sense) that were written on Newton, we may select as exhibiting the greatest insight and penetration into the master's teaching the work done by Colin Maclaurin (1698–1746), the famous professor at the University of Edinburgh. To him we owe a complete exposition of the discoveries made by Newton in the field of natural philosophy; a treatise on Algebra, which is a worthy sequel to the great *Arithmetica Universalis*; and, finally, *A Complete System of Fluxions*, which was honoured by a French translation, and which made widely known in the United Kingdom and on the Continent the methods of the infinitesimal calculus generally adopted in England. Moreover, his *Geometria Organica*, a youthful work which is generally omitted from his more original writings, takes as its point of departure the celebrated organic generation of the conics suggested by the author of the *Principia*. His memoir entitled *De Causa physica Fluxus et Refluxus Maris* won the prize offered by the University of Paris in 1740. It contains the classical work on the theory of the attraction of an ellipsoid, and is throughout informed by the Newtonian idea. It is written in the style of the ancient geometers, which Newton considered as the only style appropriate to any mathematical treatise worthy of the name.*

In his fruitful researches on the generation of plane curves, Maclaurin met with a formidable rival in William Braikenridge.† Of him nothing whatever is known beyond the fact that he was an English clergyman.

In his study of the properties of algebraic curves and in the theory of equations he found a follower capable of satisfying the most exacting demands upon his capacity. This was Edward Waring (1736–1798), one of the most acute and profound mathematicians to be found in the history of the science. In justification of this flattering estimate it is sufficient to remember that he was the first to point out the importance of the "equation of squared differences of the roots of an algebraical equation," which in the hands of Lagrange was to become a powerful weapon for the separation of the roots of a numerical equation. He discovered the method of approximating to the roots of an equation to which the name of Graefe is generally attached. He made known a fundamental theorem in the theory of numbers, discovered by his compatriot John Wilson, and afterwards rigorously established by Lagrange. And, further, he asserted "that any integer may be represented as the sum of any number of *n*th powers," and that, in particular, "any number may be expressed as the sum of nine cubes." The proof of the latter theorem was long sought for by the devotees of higher Arithmetic, and is to-day to be classed among the most brilliant of the results achieved by the irresistible logic of David Hilbert.

XI.

I have shown how the finest work of the English mathematicians of the eighteenth century came to be written.‡ On the whole, the

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* Maclaurin was also the author of a work on Geometry, which, thanks to the translation made by E. de Jonquières, was widely read on the Continent. It is a really excellent treatise on the properties of plane cubics.

† [*Exercitatio Geometrica de Descriptione Linearum Curvarum*, 1733.]

‡ There is a little exaggeration in the following lines from Matthew Arnold's essay on "The Literary Influence of Academies": "The man of genius (Newton) was continued by the English analysts of the eighteenth century, comparatively powerless and obscure followers of the renowned master. The man of intelligence (Leibniz) was continued by successors like Bernoulli, Euler, Lagrange, and Laplace, the greatest names in modern mathematics" (*Essays in Criticism*, vol. i. p. 87, Tauchnitz Edn.).

direction given to their investigations (especially by the work of Maclaurin, undertaken with the firm conviction that the best path to follow was that opened by Newton) had a deplorable effect. It completely isolated England from the enthusiastic and productive movement taking place in Germany, as shown in the work of Bernoulli and Euler; in Italy, thanks to Riccati, Fagnano, and Lagrange; and in France, from the initiative of d'Alembert and Laplace. While the simple differential notation, suggested and applied by Newton's rival, made possible, even in less gifted men, the use of the infinitesimal algorithm, the weighty kinematical and geometrical considerations characteristic of the fluxional methods rendered the discovery of further truths slow, laborious, and difficult.

In these conditions of inferiority to which English mathematicians a short time after Maclaurin voluntarily condemned themselves, we find James Ivory (1765–1842), who, liberated from the prejudices of nationality, made himself familiar with the algorithm of Leibniz, and was thus enabled to give his name to the fundamental propositions of the theory of attraction of the ellipsoid.

This deplorable state of affairs was continued for many decades. It was not till the first quarter of the nineteenth century that a group of daring young mathematicians determined to substitute the policy of "ententes cordiales" for that of "splendid isolation." They founded at Cambridge the famous Analytical Society, the origin of which has been told with loving and truthful skill by W. W. Rouse Ball.* Time forbids me to dwell upon the beneficent influence of its members. I must confine myself to mentioning the outstanding names of Woodhouse (1773–1827), Peacock (1791–1858), Babbage (1792–1871), and John Herschel (1792–1871), who were powerfully assisted in the work of reform by William Whewell (1794–1866), the famous author of the *History of the Inductive Sciences*, and by the eminent astronomer Biddell Airy (1801–1892).

To illustrate the influence exercised through the whole of Great Britain by the Analytical Society, it is sufficient to point to the brilliant constellation of stars of the first magnitude to be seen in the English mathematical firmament during the Victorian Age. That influence was full of significance and pregnant with momentous results, for there now took place between the mathematicians of England and of the Continent such an active interchange of ideas that it amounted to a fraternisation. For instance, such work as that of J. J. Sylvester (1814–1897) and A. Cayley (1821–1895) in the theory of algebraic form found devotees of the first rank in Hermite, Aronhold, Brioschi, Clebsch, Gordan, and finally David Hilbert, who placed upon it a worthy crown. The foundations of the theory of surfaces of the third order were on the one side laid by Cayley, Sylvester, and Salmon (1819–1904), and, on the other, by Steiner, Schlaefli, and Grassmann, and such researches as we find in the classical memoirs of L. Cremona and R. Sturm are like a majestic river into which its English and German tributaries have poured their fertilising flood. Finally, from Plücker's discovery of the relations deducible from the characteristics of a plane curve, Cayley was led from the analogous formulae for gauche curves to the investigation of the relations derived from the characteristics of an algebraical surface. The road now lay open to a prolific field of research into which Italy plunged with truly southern ardour, and in which she reaped a splendid harvest. In dynamics, the name of William

* v. chap. vii. of *The History of the Study of Mathematics at Cambridge* (Cambridge, 1899), and the article on "The Cambridge School of Mathematics" in the *Mathematical Gazette*, July 1912.

Rowan Hamilton (1805–1865) is closely linked with the glorious names of Lagrange and Jacobi, while in the theory of systems of rays it is connected with that of Kümmer. And who is not aware of the potent impulse given to the creation of the theory of hypercomplex numbers by Hamilton's theory of Quaternions? Another eminent English mathematician, Henry Stephen Smith (1826–1883), had such a mastery of the methods of Pure Geometry which we owe to Steiner and Chasles, and of the Theory of Numbers and the new Gaussian analysis, that he carried off the prizes offered for competition to the whole scientific world by the Universities of Paris and Berlin. Again, with George Boole (1815-1864) was initiated that important movement of thought which led to Mathematical Logic. Curiously enough, the study of this branch took no root at first in his native country. It was only after wandering about for several decades in Germany and Italy that it has recently returned to its birthplace. Finally, the names of George Green (1793–1841), G. G. Stokes (1819–1903), William Thomson (1824–1907), Clerk Maxwell (1831–1879), W. K. Clifford (1845–1879), etc., may be met with whenever we pick up a volume dealing with analysis, geometry, or mathematical physics published during the last century.

To these names many and many another might be added if my address had not already exceeded its limits, and if I did not adhere to my fixed resolution to omit those who are outside its scope. But a detailed study of the works of those I have mentioned clearly shows how the abandonment of the old position of haughty isolation adopted by the mathematicians of the eighteenth century had the best effects on the progress of the exact sciences. Hence we are justified in the name of the interests of our science in expressing the earnest hope that Great Britain will remain faithful to her policy of alliance. Not to mention its other advantages, we avoid in this way the recurrence of deplorable incidents, such as the inexplicable indifference of your country to some branches of the science which are eminently worthy of attention, such as—to take a single instance—Descriptive Geometry, which is from the points of view of both theory and practice an important discipline, and to which Great Britain would certainly make contributions worthy of the birthplace of Brook Taylor.

XII.

At the close of this rapid sketch of the mathematical glories of Great Britain, I must apologise for its unavoidable imperfections and discontinuities. The picture I have presented to you is not so much a faithful portrait as a noble mosaic in which certain important figures cannot be traced owing to the absence of numbers of its tessellae. The imperfections arise in part from the limits of my address, partly also from my own ignorance—an ignorance which I have not succeeded in dispelling in spite of the attractive nature of my subject—but partly also, it would be idle to deny, owing to the scarcity of the material at my disposal. Your country possesses a biographical literature so rich and splendid that it compares with that of any other land. It has at all times manifested a tender affection for the achievements of Greek Mathematics—and here I may be allowed to say that the persistence of that interest is proved by the weighty and masterly writings of my gifted friend Sir Thomas Heath. Yet it must be admitted that to the history of modern mathematics she has shown but slight attention.* The volumes we owe to Mr. W. W. Rouse Ball are but an

* The materials for the history of certain special epochs or theories have been collected by Halliwell, Todhunter, and Muir; and it should not be forgotten that the *Proceedings of the British Association for the Advancement of Science* contain a large number of valuable reports on the past and present of other theories and periods.

isolated peak in a boundless plain. Augustus de Morgan is a notable exception, and his example deserves to be followed. But the numerous historical articles of that man of vast culture and remarkable genius are buried in sets of volumes inaccessible to those who live on the Continent, and thus have not secured the recognition they deserve, and have not exercised the influence of which they are capable.*

The series of mathematicians here presented may be termed discontinuous. In it there are gaps which are sometimes a century across. Their history presents the attractive problem of its completion, if that be possible, or of investigating the causes which led to these sudden interruptions to mathematical research in England.

Moreover, there is a goodly company of mathematicians who have not deserved oblivion, and whose careers are but imperfectly known. The desire to know something about them is amply justified, and particularly is this the case with the personalities corresponding to the names of Pell,† Wilson, Braikenridge, Caswell, and the like.

Finally, let me draw attention to the conspicuous number of posthumous works to be found in English mathematical literature. This is a characteristic which may be attributed to a seriousness of intention on the part of the geometers of Great Britain. The moment, at once dreaded and longed for, at which the word FINIS is written was deliberately put off from day to day. Indifferent to fame, ignoring the prospects of official honours, they pursued the course of their labours, postponing the fatal moment until they were caught unawares and touched by the icy finger of death. I am, therefore, induced to ask if posterity has really extracted from the manuscripts of these great men all the valuable material that they may contain.‡ And when we think of the treasures that lie buried in the letters of Leibniz, and of the valuable material which we are at the present moment finding in the precious manuscripts of Gauss, it is only natural to hope that in your great libraries may yet be found a mine of rich and splendid ore as yet entirely unexplored. In particular, the papers of Newton, if entrusted to competent hands, may yet teach us something that it is worth our while to learn.§

Those who have the capacity of feeling the ineffable pleasure to be found in historical research will be attracted by investigations of this kind, and will provide the material for important work. It is to be hoped that the delightful Scientific Session of this Congress at the South Kensington Museum will go far to spread throughout the land a real enthusiasm for the great discoverers and their works. I hope and trust that those in these islands who can prepare themselves for such research will gird themselves in numbers to the task.

It may be that their number will be but small, and the value of the results unequal to their legitimate hopes. But requisite encouragement might very well come from a vote of the present Congress, a vote which might arouse the munificence of some private individual and

* It is most desirable that these should be collected into an organic whole. They would certainly be of the greatest value.

† There are many references to this mathematician in Rigaud's *Correspondence of Scientific Men*. Here we learn not only something of the peculiar characteristics of the man, but that he translated and wrote a commentary on the Algebra of Branker and Rhonius, 1668; that he wrote against Longomontanus (a fact known to A. von Braunmühl, *v.* vol. i. p. 58 of his *Vorlesungen über Geschichte der Trigonometrie*); that he composed a "Table of Squares"; and, finally, that he published in London, in 1650, a volume entitled *An Idea of Mathematics*, which does not seem to have deserved the silence and oblivion it has been awarded in mathematical history.

‡ Such hopes are spontaneously expressed in Vacca's letter quoted above.

§ With one accord all mathematicians are calling for a really complete edition of Newton's works containing all his unpublished writings.

establish the harmonious action of this glorious University and the powerful Scientific Societies to be found in Great Britain.

If this suggestion is adopted and the desired results are achieved, I have little doubt that new and glorious light will be reflected on your powerful and noble nation. On the day on which the *History of Mathematics in Great Britain* appears—if it be worthy of its theme—no one will rejoice more sincerely than he who, responding to your courteous invitation, and with the affection of a grateful guest, has been permitted to express with every freedom the sentiments that are shared by all those who have at heart the interests and the history of the exact sciences.

GINO LORIA.

MY LECTURE NOTES ON CALCULUS.

IN view of the recent discussions and papers in the *Gazette*, it may not be out of place for me to sketch the methods by which the Calculus is approached in my lectures. The class includes not only students qualifying for a degree in mathematics, but also others requiring a working knowledge of the use of the Calculus for the purposes of their courses in physics and chemistry.

I define a differential coefficient as follows: Let $f(x)$ be any function of x, and let x_1, x_2 be any two values of x. Then *if* the quotient

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

tends to a unique limit when x_2 and x_1 both approach a common value x, this limit is called the differential coefficient of $f(x)$.

The assumption that the limit is unique enables us to put one of the quantities x_1 or x_2 first equal to x, and we thus get particular forms of the differential coefficient as the limits of

$$\frac{f(x+h)-f(x)}{h} \text{ or } \frac{f(x)-f(x-h)}{h} \text{ or } \frac{f(x+h)-f(x-h)}{2h}.$$

Previous to giving this definition, I take the Remainder Theorem in algebra and show that if $$f(x)=A+Bx+Cx^2+Dx^3+\dots,$$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=B+C(x_2+x_1)+D(x_2^2+x_2x_1+x_1^2)+\dots,$$

which, when x_2 and x_1 are both put equal to x, shows that the quotient approaches the unique limit $B+2Cx+3Dx^2+\dots,$

so that $f(x)$ actually has a differential coefficient defined in this way.

If $y=f(x)$ and y_1, y_2 are values of y corresponding to x_1, x_2, then dy/dx is the limit of

$$\frac{y_2-y_1}{x_2-x_1},$$

when x_1 and x_2 both become equal to x. This at once identifies dy/dx with the slope of the tangent to the graph of $y=f(x)$.

It is not necessary to go into detail, but, as an example, the proof of the product rule takes the form, if $y=uv$,

$$\frac{y_2-y_1}{x_2-x_1}=\frac{u_2v_2-u_1v_1}{x_2-x_1}=\frac{u_2v_2-u_1v_2+u_1v_2-u_1v_1}{x_2-x_1}=v_2\frac{u_2-u_1}{x_2-x_1}+u_1\frac{v_2-v_1}{x_2-x_1},$$

which assumes the limiting form $=v\dfrac{du}{dx}+u\dfrac{dv}{dx}$.

Later on, as another example, the differentiation of x^x is associated with those of x^n and a^x by a similar method, *i.e.*

$$\frac{y_2-y_1}{x_2-x_1}=\frac{x_2^{x_2}-x_1^{x_2}}{x_2-x_1}+\frac{x_1^{x_2}-x_1^{x_1}}{x_2-x_1}.$$

It will be seen that the main features of this method are:

1. I do not at the present stage write $x+h$ or $x+\Delta x$ for one of the values of the variable, and thus I never get a long series of powers of Δx.

2. I make my variables approach finite limits instead of becoming equal to zero. The beginner is thus saved the initial difficulty which arises when in some places Δx is to be put equal to zero and in others it is not.

The class then learns to differentiate algebraic (not transcendental) functions and also to differentiate algebraic equations, to find tangents and normals to the conics $y^2=4ax$, $x^2/a^2+y^2/b^2=1$, $xy=c^2$, and other curves, and also to find velocities and accelerations when the space is an algebraic function of the time.

By this time the pupils are much better prepared for the use of differentials, and can now examine in what sense we are justified in substituting

$$dy=f'(x)dx \;\ldots\ldots(1) \quad \text{for} \quad \frac{dy}{dx}=f'(x). \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The meaning of the sign = in (1) is explained by simple illustrations of nearly equal quantities, 90,000,000 miles and 90,100,000 miles being more nearly equal than 2 inches and 1 inch. The criterion reduces generally to whether the ratio of the quantities does or does not differ from unity by a small fraction, and practically this leads us no further than saying that (1) has to be interpreted by means of (2). But now we are ready to start on the Integral Calculus.

G. H. BRYAN.

REVIEWS.

The Elements of Non-Euclidean Geometry. By D. M. Y. SOMMERVILLE. Pp. xvi+274. 5s. 1914. (Bell & Sons.)

It is probable that there can be no finality in the search for rigour in mathematics. Each generation finds flaws in the reasoning of its predecessor and seeks to remove them, but it has no further guarantee of success in the latter part of its task than is afforded by its own feelings of satisfaction. Just now, for instance, the tendency among mathematicians is to discredit intuition and insist on formal reasoning; yet the act of distinguishing a sound argument from an unsound one is intuition of a kind. The sound argument is the one in which nobody can "see" a flaw, and it is sound only until a new seer discerns where it also is faulty.

A by-product of the alternating process of construction and destruction is the subject of Non-Euclidean Geometry, which has arisen, as everyone knows, from the dissatisfaction of posterity with the form in which Euclid left the theory of parallels. To the practical man who prefers the seen to the unseen the subject is foolishness, but to one interested in the achievements of the human mind it is not the least beautiful among the many ornaments of the mathematical edifice. Its chief attraction is, of course, for the keen geometer and the mathematical logician, but it cannot safely be ignored by the elementary teacher, the student of higher analysis, or the worker in the region of dynamics and astronomy.

Dr. Sommerville's book begins with a history of the various attempts to prove Euclid's postulate concerning parallels, and follows this up with expositions of the hyperbolic and elliptic theories. The most interesting part is the establishment of functional and numerical relations (it is a pity that the author calls them relationships), that is to say, of trigonometry. In the one case this is treated in the manner of pure geometry and in the other by means of a differential equation. This intentional variety of presentation adds to the interest of the book and was quite desirable in a course of lectures at the Edinburgh Colloquium, such as that on which this book is founded. It has, however, the disadvantage of leaving the reader uncertain as to the foundation on which he is working. To obviate this the author refers to the works of Coolidge and others for a systematic treatment based on axioms.

There is a chapter on the philosophical bearing of the subject, and the rest of the work consists of various developments relating among other things to the use

of coordinates, properties of triangles, systems of circles, inversion and other transformations, and conics. One chapter is devoted to the different representations of Non-Euclidean space in Euclidean, with references to the generalised idea of distance as the logarithm of a cross-ratio, and to the theory of surfaces on which Gauss' measure of curvature is constant.

The author has given us a useful and stimulating book, and has not forgotten to add examples after most of the chapters. He has been well served by the Glasgow University Press. There are few slips. One is at the middle of p. 61 where "perpendicular" is written for "parallel," and another is the statement on p. 112, that if $AB=CD$, and $\angle ABD=\angle DCA$, then AB is paratactic to CD.

A. C. Dixon.

1. **New Concrete Arithmetic.** By C. Pendlebury, M.A., and H. Leather, M.A. In Five parts, 4d. each. 60-70 pp. each part. (Bell.)

2. **New Concrete Practical Arithmetic Tests.** Edited by J. L. Martin. In Five parts, 4d. each. 64 pp. each. (Harrap.)

1. An excellent set for primary school work. Each rule is first presented in concrete form, abstract processes are introduced gradually as familiarity with concrete examples is secured. The authors do not, however, seem to insist enough on the fact that the abstract processes are the fundamental processes, and that the concrete forms are only applications. This may prove a stumbling block to students when they later study Practical Mathematics, or higher work in Pure Mathematics, and are confronted with symbolic methods. One of the great points about this series is "the economy of time and effort," and to a greater extent the gain in intelligent handling, secured by the simultaneous teaching of inverse or Complementary Processes. The series deserves a wide sale.

2. This series, animated by the same idea as the former, consists, however, only of test papers. As such they should save the teacher considerable time and trouble; but from the point of view of educational gain they cannot be compared with the former series, which not only supplies the examples but directs the teaching. The examples, however, seem to be both suitable and well chosen; nearly all are of a problem type, but not catchy. They demand thought, but not more thought than one has a right to expect from the average boy.

Percentage Trigonometry; Elementary Treatise for Schools, etc. By J. C. Fergusson. Pp. 150. Price 3s. 6d. (Longmans.)

This exceedingly interesting text-book is an attempt to give an elementary rendering of the author's invention of a new method of figuring a protractor, so as to render the problems of Plane Trigonometry a matter of mere arithmetic of a most simple kind.

As a school text, I cannot see it as anything but a failure. It suffers greatly from lack of that conciseness and preciseness which is nowadays considered (and justly so) to be a *sine quâ non*: there is also a want of orderly method; and there are no exercises for the student to work. As an instance of obscure phrasing, take the author's Rule I., considered as a *Rule* to set before a beginner:

"In a right-angled triangle, when the perpendicular is divided by the base and the base is divided by the perpendicular, the ratios between these two lines are found, and the figures of the ratios serve as the correct numbering of the percentage angles formed at the ends of the hypotenuse. Moreover, when any one of the sides and any angle, other than the right angle, are known, then the other sides and the other angle are apparent."

This is as bad as the extended definition (?) of multiplication given in texts on Algebra: "To multiply a by b, you do to a what you do to unity to get b"!

His second Rule suffers by having a comma misplaced; its real meaning is, of course,

$$\sec\theta=\sqrt{1+\tan^2\theta},$$

but it reads, at present, $\sec\theta=\sqrt{1}+\tan^2\theta$.

All these points could have been avoided and a really excellent little exposition of the percentage system given, if Mr. Fergusson had collaborated with an expert writer of text-books for schools. The pity of it is that the method is an extremely ingenious and useful one, though I am far from being convinced that all the advantages the author claims for it are substantiated by the real facts.

There is certainly the advantage of doing without tables. But I understand that the large treatise includes tables of logarithms. If tables have to be used at all, I do not see that this point counts for much.

As a compass-card, I do not see how it can be read to greater accuracy than ½ %. This is not three-significant figure accuracy; so that the cosine circle reading to five figures is not only a pretence, but a dangerous mislead. Of course, it depends on the diameter of the dial for one thing, but to a much greater extent on the coarseness or fineness of the lubber's line.

As a theodolite, there is a distinct advantage in the fact that a difference of percentage angle of say 5 %, as marked on the dial-ring, subtends on the perpendicular a difference of level which is the same all over the perpendicular from 0 % to 100 %, thus enabling the base-line to be calculated at once. But when the difference of percentage angle is obtained by two sights this advantage disappears, unless the sight-angles are small. Here again we have the question of significant-figure accuracy. The ordinary small theodolite with a 5 in. to 6 in. radius circle usually reads *by means of verniers* to half a minute accuracy; the percentage angle marks for 16° 41′ 57″ (30 %) and 17° 13′ 24″ (31 %) would be less than 0·01 in. apart, and since the divisions are unequally spaced, the equivalent to a vernier reading must be obtained by means of complicated mechanism liable to get out of adjustment. (This is suggested by the figure of the theodolite given at the end of the book. I confess I am unacquainted personally with the instrument itself.) I do not see therefore how more than three-figure accuracy can be obtained. To this degree of accuracy there is no possible doubt as to the extreme rapidity and ease with which all kinds of survey work can be accomplished without recourse to tables (see the report, inserted at end of the book, by Prof. Heath, who had the opportunity of testing the instrument).

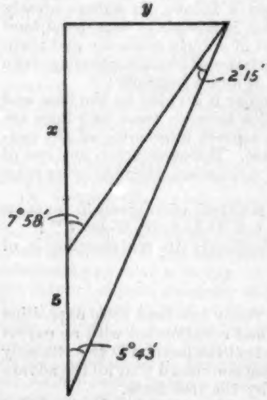
Here take as an example the author's working of Ex. XV., where he gives the quotient of 400 ÷ 5·22 (three-figure accuracy) as 76·398 (five-figure accuracy). The "correct-to-three figures" answer 76·4 would, however, have been quite good enough for this problem.

As a method for scouting, it is useless. Take the worked-out example on page 136 (shortened):

"Two telegraph poles directly in front of a scout, known to be 80 yds. apart, subtend an angle of 5½ %" (*i.e.* about 3°). "Determine the range."

How is he going to measure this angle? A scout would not carry about a card greater than a 4-inch square or a compass card 4 inches in diameter. With this he would not be sure of the angle to within ½ %. Now 5 % and 6 % would give the range as 1333 yds. and 1600 yds. respectively. Why then give 1455 yds., which states that the distance is known to half a yard?

I do not know how army scouts would work; they have their own methods; and a good scout would at once *guess* the distance as 1500 yds.; but a boy-scout would point his pole towards the middle point between the posts, place his eye at one end and move a half-penny along the pole until it "covered" the distance between the posts. In the present problem he would find the distance between his eye and the half-penny to be about 18 to 18½ inches. Hence, range is between 18 × 80 = 1440 yds. and 18½ × 80 = 1480 yds., for he knows a half-penny is exactly one inch across. All this is supposing he could see two telegraph posts against any background but the sky at 1500 yds.



Considering the other two worked-out problems, since one cannot imagine the author purposely giving misleading methods, one must think he is unaware of the best methods in ordinary plane trigonometry.

For Problem 2, one always teaches to a beginner a method almost identical with his method by means of his Equation 3 (see any text on *Numerical* Trigonometry). I was taught it twenty-eight years ago! It certainly requires tables, and is not so short as Mr. Fergusson's, but takes no more than four minutes to work out;

the figures required are those below. It, however, has the merit that any degree of accuracy can be obtained.

$$\frac{x+5}{y}=\cot 5^\circ 43'=9\cdot98931,$$

$$\frac{x}{y}=\cot 7^\circ 58'=7\cdot14553\,;$$

$$\therefore\ \frac{5}{y}=2\cdot84378\,;$$

$$\therefore\ y=\frac{5}{2\cdot84378}=1\cdot76,$$

$$x=1\cdot758\times 7\cdot1455=12\cdot56.$$

$$\begin{array}{r} 1\cdot758 \\ 2,\cdot8,4,3,8\,)\,\overline{5\quad\;\;} \\ \overline{2156} \\ \overline{166} \\ 24 \end{array} \qquad \begin{array}{r} 7,\cdot1,4,5,5 \\ 1\cdot758 \\ \hline 7\cdot146 \\ 5\cdot002 \\ 357 \\ 57 \\ \hline 12562 \end{array}$$

The approximation in the margin could be done rapidly with a slide-rule.

He says, too, that "the problem would take too much time to work out by natural sines and cosines." It takes exactly seven to eight minutes, inclusive of making out the formulae and taking out the extracts from sine tables. The figures are:

$$\frac{5\times\sin 5^\circ 43'\cos 7^\circ 58'}{\sin 2^\circ 15'},\quad \frac{5\times\sin 5^\circ 43'\sin 7^\circ 58'}{\sin 2^\circ 15'},$$

$$\frac{5\times 0\cdot09961\times 0\cdot99035}{0\cdot03926},\quad \frac{5\times 0\cdot09961\times 0\cdot13860}{0\cdot03926}.$$

$$\begin{array}{r} 39,2,6\,)\,\underline{49805}\,(\,12\cdot68 \\ \underline{1055}\qquad\qquad \\ \underline{270}\qquad\qquad \\ 35\qquad\qquad \end{array} \qquad \begin{array}{r} 12\cdot68 \\ \cdot99035 \\ \hline 11\cdot412 \\ 1\cdot141 \\ 4 \\ 1 \\ \hline \underline{12\cdot56} \end{array} \qquad \begin{array}{r} 12\cdot68 \\ \cdot1386 \\ \hline 1\cdot268 \\ 380 \\ 101 \\ 7 \\ \hline \underline{1\cdot76} \end{array}$$

Taking the traverse example, the comparison is most unfair. Using ordinary traverse tables, we have (*working from Chambers to degrees only*):

Course.	Angle.	Northing.	Easting.
1000	2°	999	35
500	22°	187	464
300	11°	57	−295
		1243	204

Looking through the traverse tables, with attention fixed on 124·3 in Lat. col., we find the nearest departure to 20·4 is 19·7 for 9° with Dist. 126.

Therefore closure line is 1260 yds., and the angle a little more than 9° W. of S.

The author gives a method, taking two and a half pages, to compare with his own method taking half a page! A set of traverse tables up to distance 300 occupies only forty-five pages in Chambers. If this is objected to as cumbrous, I could get all that is required on two sides of a postcard, the only arithmetic necessary being multiplication by 10 and addition. For traverse work Fergusson's method can claim no advantage; if anything the advantage lies with the ordinary method.

In short, in my opinion, whilst it is a "master method" for all rough work, except traverse work, I cannot see that it is comparable with the ordinary methods for delicate work such as ordnance survey, or for use at sea for long voyages (demanding spherical trigonometry), certainly not for "great circle sailing," nor can I see its easy application to the sextant for observation purposes.

J. M. Child.

NOTICE.

"A LEGEND of Level-Land" (based on Dr. Abbott's "Flatland") was acted at the Haberdashers' Girls' School, Acton, in June 1913 (v. *Math. Gaz.*, Jan. 1914). Applications for information about the play have been received from many quarters. Messrs. Bell & Sons are prepared to publish the "Legend" if the sale of a sufficient number of copies can be guaranteed. Miss Brown will be glad to have, from teachers who propose to purchase the play, an estimate of the number of copies they are likely to require. Letters may be addressed to the school, or to 50 Hillcrest Rd., Acton, W. It may be added that "Flatland" has long been out of print, and is unlikely to be reprinted.

THE LIBRARY.

THE Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Wanted by purchase or exchange:

1 or 2 copies of *Gazette* No. 2 (very important).
1 or 2 " " No. 8.
2 or 3 copies of Annual Report No. 11 (very important).
1 or 2 " " Nos. 10, 12 (very important).
1 copy " Nos. 1, 2.

BOOKS, ETC., RECEIVED.

A Foundational Study in the Pedagogy of Arithmetic. By H. B. HOWELL. Pp. xii+328. 5s. 6d. net. 1914. (Macmillan Co.)

Geometry of Four Dimensions. By H. P. MANNING. Pp. x+348. 8s. 6d. net. 1914. (Macmillan Co.)

Annals of Mathematics. Edited by O. STONE and others. Series II. Vol. 16. No. 1. Sept. 1914. 2$ per vol. (Lancaster, Pa.)

Invariantive Characterization of some Linear Associative Algebras. O. G. HAZLETT. *On the Theory of n-Lines.* L. P. COPELAND. *Plane Curves with Consecutive Double Points.* F. H. SHARPE and C. F. CRAIG. *The Effect of Radiation upon a small Particle revolving about Jupiter.* T. H. BROWN. *Non-Homogeneous Linear Equations in infinitely many Unknowns.* A. J. PELL. *The Inscribed and Circumscribed Squares of a Quadrilateral, and their Significance in Kinematic Geometry.* C. M. HERBERT. *A Note on Symmetric Matrices.* G. A. BLISS.

A First Book of Practical Mathematics. By T. S. USHERWOOD and C. J. A. TRIMBLE. Pp. 182. 1s. 6d. 1913. (Macmillan.)

Practical Geometry and Graphics for Advanced Students. By J. HARRISON and G. A. BAXANDALL. Pp. xiv+677. 6s. 1913. (Macmillan.)

The Twisted Cubic, with some account of the Metrical Properties of the Cubical Hyperbola. By P. W. WOOD. No. 14. Cambridge Tracts. Pp. 78. 2s. 6d. net. 1913. (Cam. Univ. Press.)

An Introduction to the Algebra of Quantics. By E. B. ELLIOTT, F.R.S. 2nd Edition. Pp. xvi+416. 15s. net. 1913. (Clarendon Press.)

Elements of Geometry. By S. BARNARD and J. M. CHILD. Parts I.-VI. Pp. ix+465+pp. xxxviii. Exam. Papers+Answers. 4s. 6d. 1914. (Macmillan.)

Essays on the Life and Work of Newton. By AUGUSTUS DE MORGAN. Edited with Notes and Appendices by P. E. B. JOURDAIN. Pp. xi+198. 5s. net. 1914. (Open Court Company.)

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